SENIOR PROJECT

Column Shape Optimization for Maximum Elastic Buckling Capacity

Suneenut Suewongprayoon[1], Supin Sartbumrung[2], Tosporn Prasertsri[3*] and Jaroon Rungamornrat[4]**

[1,2,3,4]Applied Mechanics and Structures Research Unit, Department of Civil Engineering,
Faculty of Engineering, Chulalongkorn University, Bangkok 10330
E-mail: [1]ssuewongprayoon@gmail.com, [2]supinsbr@gmail.com, [3]Jaroon.r@chula.ac.th, [4]tospond.p@hotmail.com
[*]Research Assistant, [**]Senior Project Advisor

Abstract

This senior project presents a gradient-descent, finite-element-based procedure for the shape optimization of column to attain the maximum elastic buckling load subjected to the volumetric constraint. Linearized Euler-Bernoulli beam theory is adopted to form the equation governing the flexural buckling of the column and a finite element method with hermite element shape functions is implemented to determine the corresponding numerical solutions. The constant, linear, and quadratic element shape functions are employed to discretize the cross-sectional area of the column. The least eigenvalue and the corresponding eigenvector of the resulting eigen-system for any column shape are efficiently calculated using power method together with Rayleigh quotient. Standard differentiations are applied to obtain the information essential for the gradient descent method. A selected set of results is then reported to demonstrate the convergence and accuracy of numerical solutions.

Keywords: Shape optimization; Elastic buckling; Columns; Finite element method; Gradient descent method

1. Introduction

1.1 Motivation and significance

Despite optimization was invented for several years, it is still attractive for many scientists and analysts. optimization is the selection of the best element from some set of available choices. In the civil engineering problem, the solution of optimization can be the minimum or maximum value of the objective functions. In this paper, we aim to focus on applying optimization for the design column. Column shape optimization has various advantages over conventional column design. For example,column optimization reduces the cost of a project. The optimization problem can be divided into two methods depending on the solution. There are exact solutions and numerical solutions.Numerical solution provides the approximate answers of the optimization. But this method can be used in complex problems. Although, previous papers had solved this problem by the simple method like using Hencky bar-chain model. But the disadvantages of Hencky bar-chain model are (1) need to separate thousands of elements along the column to give the nearly exact solution and (2) to assume column as rigid columns. In this paper, we improve and develop to provide more simple and powerful method, fast convergence, high accuracy by using finite elements.

1.2 Background and review

1.2.1 Exact solutions

Keller [1] formulated the problem to determine the column shape with a maximum buckling load buy using isoperimetric inequality for a certain eigenvalue problem. The critical buckling load of the optimal-shaped column was larger by 61.2% compared to the column with a cylindrical shape section. Then, Tadjbakhsh and Keller [2] had solved the optimization problems by isoperimetric inequalities for eigenvalues of second-order ordinary differential equations with various boundary conditions. The result showed that the critical buckling load is largest for the case with the hinged-hinged condition. The results were also expressed as isoperimetric inequalities for eigenvalues of second-order ordinary differential equations with various boundary conditions

1.2.2 Numerical solution

Maalawi [3] focused on a buckling optimization of flexible columns. The research showed that the model which was not restricted to the cross-section can provide a higher critical buckling load compared to the literature. Krishna and Ram [4] used the discrete link-spring model with clamped or pinned at one end and spring-supported at the other end. The model is used to verify the results obtained by Tadjbakhsh and Keller [2]. The solutions obtained using a one-parameter iterative loop. Wang et al. [5] applied the Hencky bar-chain model (HBM) for the buckling and vibration analyses of non-uniform beams resting on a partial variable elastic foundation. The study [5] proposed an approximate model for optimizing Bernoulli columns using a combination of HBM techniques and genetic algorithms. Wang et al. [6] pointed out that the approach proposed by Krishna and Ram [5] may not be applicable in practice since a small discretization may affect the accuracy of the calculated buckling load. Wang et al. [7] presented the method for optimization of column resisting buckling when both compressive concentrated and distributed axial load.

1.3 Research objective

The key objective of this senior project is to implement a gradient descent, finite element based solution procedure for optimizing the column shape to attain the maximum elastic buckling load subjected to a volumetric constraint.

1.4 Scope of work

The present study is carried out specifically for a perfectly straight column with a circular cross section and fixed-free end conditions. The shear and axial deformations are fully negligible and the elastic flexural buckling is considered as the only mode of failure of the column.

1.5 Methodology and research procedure

The methodology and research procedure employed in the present study can be briefly summarized below. First, using Euler-Bernoulli beam theory to formulate the governing equations. Next, using finite element procedure to transform the governing into a linear eigen-system. Then, power method with Rayleigh quotient to determine the least eigenvalue and corresponding eigenvector. After that, The optimal profile of the cross-sectional area is achieved via the gradient descent method. Last, the proposed numerical procedure is implemented within the framework of MATLAB and then verified with available benchmark solutions

1.6 Anticipated outcome and contribution

This proposed is to use fast convergence, high accuracy method by adapting finite element method to find the optimal shape of the column.

2. Problem Formulation

2.1 Problem description

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Consider a cantilever beam of length l subjected to a compression force P at the free end. The column is made of a homogeneous, isotropic, linearly elastic material of Young's modulus E and the cross section is of a circular shape with the area A=A(x) where x is a selected coordinate along the axis of the column with x=0 and x=L denoting the clamped and free ends, respectively. Let P_{cr} denote the buckling load of this particular column (i.e., the least compression force P at the bifurcation equilibrium states). The problem statement, here, is to determine the maximum P_{cr} subjected to the following volumetric constraint:

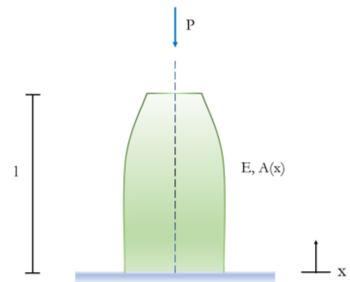


Fig 1. Cantilever column considered in the current research

$$\int_0^l A(x)dx = V_0 \tag{1}$$

where V_0 is a given volume of the constituting material.

2.2 Governing equations

At the onset of the flexural buckling (i.e., $P = P_{cr}$), the shear force V, the bending moment M, the curvature $\varkappa$, the rotation θ, and the deflection v at any cross section located at the coordinate x are related by

$$\frac{dV}{dx}=0, \quad \frac{dM}{dx}+P_{cr}\frac{dv}{dx}=V \tag{2}$$

$$M=EI\varkappa=\frac{EA^2}{4\pi}\varkappa \tag{3}$$

$$\varkappa=\frac{d\theta}{dx}, \quad \theta=\frac{dv}{dx} \tag{4}$$

where $I=A^2/4\pi$ denotes the moment of inertia of the circular cross section. Combining Eqs. (2)-(4) leads to the linear, homogeneous, fourth order, ordinary differential equation governing the deflection $v = v(x)$:

$$\frac{d^2}{dx^2}\left(\frac{EA^2}{4\pi}\frac{d^2v}{dx^2}\right)+P_{cr}\frac{d^2v}{dx^2}=0 \tag{5}$$

In addition to satisfying the governing equation (5), the deflection $v = v(x)$ must also satisfy the following essential and natural boundary conditions at both ends as listed below

$$v(0)=0, \quad \theta(0)=\frac{dv}{dx}(0)=0 \tag{6}$$

$$M(L) = \frac{EA^2}{4\pi}\frac{d^2v}{dx^2}(L) = 0; \quad V(L) = \frac{d}{dx}\left(\frac{EA^2}{4\pi}\frac{d^2v}{dx^2}\right)(L) + P_{cr}\frac{dv}{dx}(L) = 0 \tag{7}$$

Upon introducing the following normalizations $\overline{x} = x / l$, $\overline{v} = v / l$, $\overline{A} = A / A_0$, $\overline{P} = 4\pi P_{cr} l^2 / EA_0^2$, the strong statement of the buckling problem is to Find the least normalized axial force $\overline{P}$ (or the normalized buckling load) such that there exists a nontrivial function $\overline{v} \in C^4$ (i.e., $\overline{v} \neq 0$) satisfying

$$\frac{d^2}{d\overline{x}^2}\left(\overline{A}^2\frac{d^2\overline{v}}{d\overline{x}^2}\right) + \overline{P}\frac{d^2\overline{v}}{d\overline{x}^2} = 0 \tag{8}$$

$$\overline{v}(0) = 0, \quad \overline{v}'(0) = 0 \tag{9}$$

$$\overline{v}''(1) = 0; \quad (\overline{A}^2\overline{v}'')'(1) + \overline{P}\overline{v}'(1) = 0 \tag{10}$$

The solution. By applying the standard weighted residual technique together with the integration-by-part procedure, the above strong statement can be replaced by the following equivalent weak statement: Find the normalized buckling load $\overline{P}$ such that there exists a nontrivial function $\overline{v} \in H_0^2$ such that

$$\int_0^1 \overline{A}^2 \frac{d^2\overline{w}}{d\overline{x}^2}\frac{d^2\overline{v}}{d\overline{x}^2} d\overline{x} - \overline{P}\int_0^1 \frac{d\overline{w}}{d\overline{x}}\frac{d\overline{v}}{d\overline{x}} d\overline{x} = 0 \tag{11}$$

for any test function $\overline{w} \in H_0^2$ where $H_0^2 = \{f : f(0) = 0 \text{ Ù } f'(0) = 0 \text{ Ù } f'' \text{ is square integrable}\}$.

Now, the mathematical statement of the optimization problem described in the previous section is to Find the maximum value of the normalized buckling load $\overline{P}$ subjected to the following volumetric constraint

$$\int_0^1 \overline{A}(x)d\overline{x} = \overline{V}_0 \tag{12}$$

where $\overline{V}_0 = V_0 / A_0 l$.

3. Numerical implementation

3.1. Discretization

To determine the normalized buckling load $\overline{P}$ for any given function $\overline{A}(x)$ satisfying the constraint (12), a standard finite element technique is adopted together with the power method to determine the least eigenvalue. First, the column occupying the interval $[0,1]$ in the normalized space is partitioned into n finite elements such that $[0,1] = \bigcup_{e=1,n} \Omega_e$ where $\Omega_e = [\overline{x}_{e-1}, \overline{x}_e]$, $\overline{x}_0 = 0$ and $\overline{x}_n = 1$. The trial and test functions for any generic element Ω_e can be approximated by

$$\overline{v}^e(\overline{x}^e) = (N^e)^T\overline{v}^e, \quad \overline{w}^e(\overline{x}^e) = (N^e)^T w^e \tag{13}$$

where $\overline{x}^e = \overline{x} - \overline{x}_{e-1}$ denotes the local coordinate of the element Ω_e; $\overline{v}^e = \{\overline{v}_1^e \quad \theta_1^e \quad \overline{v}_2^e \quad \theta_2^e\}^T$ is a vector containing the normalized end displacements $\overline{v}_1^e, \overline{v}_2^e$ and the end rotations θ_1^e, θ_2^e; w^e is an arbitrary vector; and $N = N(\overline{x}^e)$ is a vector containing the following standard hermite shape functions where $\overline{h}^e$ denotes the length of the element Ω_e. The normalized area $\overline{A}$ of any cross section of the element Ω_e can also be approximated by

$$\overline{A}^e(\overline{x}^e) = \sum_{i=1}^{m} \varphi_i^e(\overline{x}^e)\overline{A}_i^e \tag{14}$$

where *m* denotes the number of interpolation points within the element Ω_e; $\overline{A}_1^e, \overline{A}_2^e, ..., \overline{A}_m^e$ are normalized area of the cross section at interpolation points; and $\varphi_1^e, \varphi_2^e, ..., \varphi_m^e$ are standard interpolation functions defined over the element Ω_e. Examples of $\varphi_1^e, \varphi_2^e, ..., \varphi_m^e$ for m=1,2,3 are shown in Table 1.

By substituting the approximation (13) and (15) over the finite element mesh $\bigcup_{e=1,n} \Omega_e$ into the weak form (11), it leads to a system of homogeneous, linear algebraic equations governing the approximate normalized buckling load $\overline{P}$:

$$(K - \overline{P}M)U = 0 \tag{15}$$

where U denotes a vector containing all degrees of freedom of the discretized column at the onset of the buckling, and K, M are matrices defined by

$$K = \sum_{e=1}^{n} k^e, \quad k^e = \sum_{i=1}^{m}\sum_{j=1}^{m} k_{ij}^e \overline{A}_i^e \overline{A}_j^e, \quad k_{ij}^e = \int_{\Omega_e} \varphi_i^e \varphi_j^e C^e (C^e)^T d\overline{x}^e \tag{16}$$

$$M = \sum_{e=1}^{n} m^e, \quad m^e = \int_{\Omega_e} B^e (B^e)^T d\overline{x}^e \tag{17}$$

in which $B^e = dN^e/d\overline{x}^e$, $C^e = dB^e/d\overline{x}^e$, and the summations appearing in (16) and (17) imply the direct assembly of element contribution via the standard procedure. The explicit expression for m^e can be readily obtained, from the direct integration, as

The explicit expression k^e can also be constructed and results are given in Table 2 for m=1, 2 and Table 3 for m=3. Upon the interpolation (14), the volumetric constraint (12) becomes

$$F^T\overline{A} = \overline{V}_0 \tag{18}$$

where $\overline{A}$ is a vector storing all different $\overline{A}_j^e$, e=1,2,...,n; j=1,2,...,m and F is a vector defined by

$$F = \sum_{e=1}^{n} f^e, \quad f^e = \int_{\Omega_e} \varphi^e(\overline{x}^e) d\overline{x}^e, \quad \varphi^e = \{\varphi_1^e \ \varphi_2^e \ ... \ \varphi_m^e\} \tag{19}$$

where, again, the summation appearing in (19) is carried out via the direct assembly procedure. The explicit expressions of f^e and F for m=1,2,3 are given in Table 4. Now, the statement of the discretized problem is to maximize the buckling load $\overline{P}$ satisfying Eq. (15) and subjected to the volumetric constraint (18).

3.2 Solution procedure

To determine the buckling load $\overline{P}$ of the Eigen-system (16) for any given data of the normalized area $\overline{A}$, an iterative procedure based upon the standard power method together with Rayleigh quotient. To maximize the buckling load $\overline{P}$ under the volumetric constraint (20), a standard gradient descent method is implemented. First, the gradient of $\overline{P}$ with respect to each entry of $\bar{A}$ must be obtained. From the volumetric constraint (20), the first entry of $\bar{A}$ can be expressed in terms of the remaining entries by

$$\bar{A}_1=\frac{\bar{V}_0}{F_1}-\left(\frac{F_2}{F_1}\bar{A}_2+\frac{F_3}{F_1}\bar{A}_3+...+\frac{F_p}{F_1}\bar{A}_N\right) \tag{20}$$

where N is the number of entries in $\bar{A}$; $\bar{A}_i$ denotes the *i*th entry of the vector $\bar{A}$; and F_i denotes the *i*th entry of vector F . In this sense, all the entries $\bar{A}_i$ $(i=2,3,...,N)$ can be considered independent whereas the first entry $\bar{A}_1$ is dependent on the others. By taking a partial derivative of (20) with respect to $\bar{A}_i$ $(i=2,3,...,N)$, it gives rise to

$$\frac{\partial\bar{A}_1}{\partial\bar{A}_i}=-\frac{F_i}{F_1}, \quad i=2,3,...,N \tag{21}$$

It is worth noting that the buckled shape U corresponding to the buckling load $\bar{P}$ is unique only up to the scaling magnitude (i.e., if U is the buckled shape, then αU is always the buckled shape for any α≠0). As a result, one of its entries can be set, without loss, to a unity. By taking a partial derivative of the governing equation (15) with respect to $\bar{A}_i$ $(i=2,3,...,N)$, it leads to

$$(K-\bar{P}M)\frac{\partial U}{\partial\bar{A}_i}-\frac{\partial\bar{P}}{\partial\bar{A}_i}MU=-\frac{\partial K}{\partial\bar{A}_i}U \tag{22}$$

The system (22) is sufficient for determining the gradients $\partial U/\partial\bar{A}_i$ and $\partial\bar{P}/\partial\bar{A}_i$ for i=2,3,...,N once the normalized buckling load $\bar{P}$ and the corresponding buckled shape U are obtained from the power method. The gradient $\partial K/\partial\bar{A}_i$ for i=2,3,...,N can be obtained explicitly as

$$\frac{\partial K}{\partial\bar{A}_i}=\sum_{e=1}^{n}\frac{\partial k^e}{\partial\bar{A}_i} \tag{23}$$

$$\frac{\partial k^e}{\partial\bar{A}_i}=\sum_{p=1}^{m}\sum_{q=1}^{m}k^e_{pq}\left(\bar{A}^e_q\frac{\partial\bar{A}^e_p}{\partial\bar{A}_i}+\bar{A}^e_p\frac{\partial\bar{A}^e_q}{\partial\bar{A}_i}\right) \tag{24}$$

The explicit expressions of $\partial K/\partial\bar{A}_i$ for i=2,3,...,N and m=1,2,3 are given in Table 5 and 6. The iterative procedure for the gradient descent method.

4. Numerical Results

4.1 Verification

Tadjbakhsh and Keller[2] solved the exact solution of optimal shape by using isoperimetric inequalities for eigenvalues of second-order ordinary differential equations for cantilever column.

The value of the maximum buckling load and the area of element is reported in Table 1. From the results, we can conclude that the area at the fixed end is maximum value but the area at the free end is a minimum value. It can be seen that with sufficiently large n, the maximum buckling load $\bar{P}$ is increased and buckling load ratio $\bar{P}/P_{uniform}$ converging to the optimal solution of 4/3 for both modes of interpolation function as can see in Figure 1(a). $P_{uniform}=\frac{\pi^2}{4}$ is the critical buckling load that associated with the state of neutral equilibrium. Using m=2 lead the buckling load faster convergent than m=1. Figure 1(b) can be concluded that in case using small number of elements, using m=2 has significantly less %error than using m=1. On the other hand, using over 128 elements for case m=1 is very close to value of %error by case m=2.

Table 1. Values of the maximum buckling load and the area at the fixed end A(0) and the free end A(1) for m=1,2

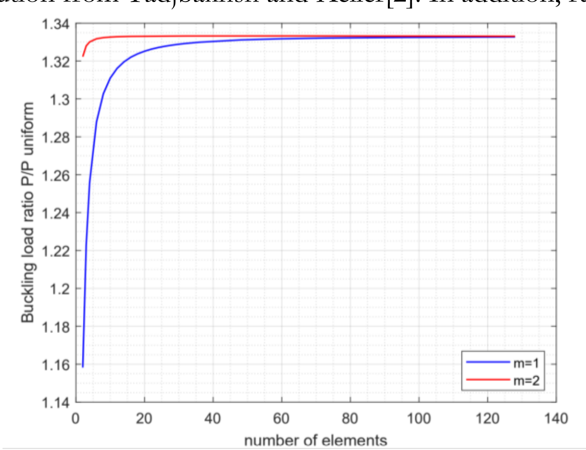
n	h_e	m=1					m=2				
		$\bar{P}$	$\frac{\bar{P}}{P_{uniform}}$	A(0)	A(1)	Time (sec)	$\bar{P}$	$\frac{\bar{P}}{P_{uniform}}$	A(0)	A(1)	Time (sec)
2	1/2	2.8578	1.1582	1.2136	0.7864	0.00635	3.2626	1.3223	1.3486	0.2844	0.02857
4	1/4	3.0989	1.2559	1.2915	0.5596	0.01663	3.2818	1.3301	1.3416	0.1674	0.04734
8	1/8	3.2143	1.3027	1.3199	0.3763	0.04843	3.2874	1.3323	1.3366	0.1041	0.11834
16	1/16	3.2619	1.3220	1.3306	0.2457	0.36113	3.289	1.3330	1.3366	0.0694	0.76538
32	1/32	3.2798	1.3293	1.3351	0.1581	2.61617	3.2895	1.3332	1.3376	0.0512	5.31556
64	1/64	3.2862	1.3318	1.3378	0.1015	30.169	3.2896	1.3332	1.3385	0.044	67.7449
128	1/128	3.2883	1.3327	1.3399	0.0664	746.192	3.2892	1.3331	1.3396	0.0673	946.572

Figure 2(a) shows the development of the variation of cross-sectional areas in case using m=1. It can be concluded that using more 128 elements provides the value that are closely exact value. Figure 2(b) shows the development of the variation of cross-sectional areas in case using m=2. It can be concluded that using more 32 elements provides the value that are closely exact value.

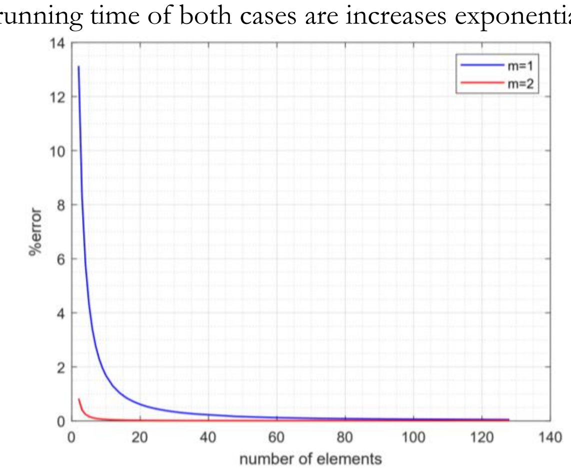
Running time for Matlab program as can be seen from Figure 3, It is concluded that running time of both cases are increases exponentially. In any number of elements, m=2 takes time longer than m=1.

5. Conclusion and Remark

This research presented a simple, fast convergence, and high accuracy for the optimization of column against buckling load. The key procedures are obtained by adapting finite element and gradient descent method. The presents solution is verified by exact result by Tadjbakhsh and Keller[2]. The results have been compared with available benchmarks from the work of Tadjbakhsh and Josephin [2]. this study confirms that the present method can faster convergence than the previous methods. In addition, the results show that the number of interpolation points (m) effect in convergent of the buckling load. That, two points of interpolation lead the convergent of buckling load under lower element than one point of interpolation. Moreover, this paper present that the optimal shape from this paper provide nearby and high accuracy exact solution from Tadjbakhsh and Keller[2]. In addition, running time of both cases are increases exponentially.

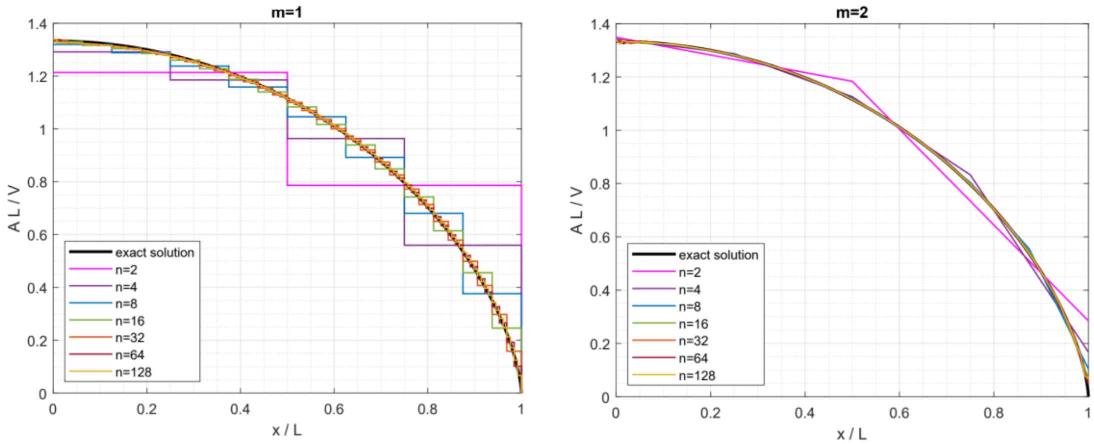


(a)



(b)

Fig 1. (a) Buckling load ratio $\bar{P}/P$ uniform versus the number of elements and (b) Percent of error versus the number of elements



(a) (b)

Fig 2. (a) Variation of cross-sectional compared with exact solution area for m=1 and (b) for m=2

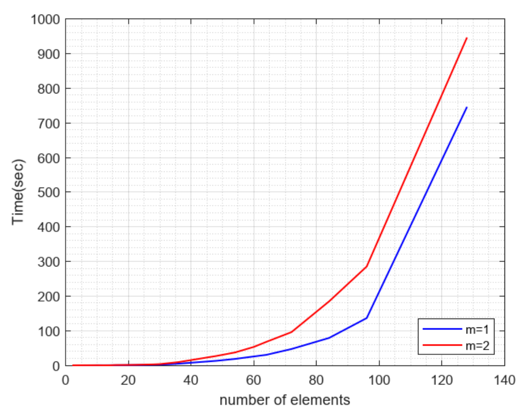


Fig 3. Running time versus number of elements for m=1 and m=2

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